RATIONAL PERIOD FUNCTIONS ON THE HECKE GROUPS *

Wendell Culp-Ressler
Department of Mathematics
Franklin & Marshall College
Lancaster, PA 17604-3003

July 11, 2000

Abstract

We describe rational period functions on the Hecke groups and characterize the ones whose poles satisfy a certain symmetry. This generalizes part of the characterization of rational period functions on the modular group, which is one of the Hecke groups.

1 INTRODUCTION

Marvin Knopp [11] introduced the idea of a rational period function (RPF) for an automorphic integral of weight $2k$ ($k \in \mathbb{Z}$) on Γ, where Γ is any Fuchsian group acting on the upper half-plane.

Knopp [11, 12], Hawkins [6], and Choie and Parson [2, 3] took steps toward an explicit characterization of RPFs on the modular group $\Gamma(1)$. Ash [1] used cohomological techniques to provide a characterization, after which Choie and Zagier [4] and, independently, Parson [14] provided an explicit characterization of the RPFs on $\Gamma(1)$. The explicit characterizations use negative continued fractions to establish a connection between the poles of RPFs and binary quadratic forms.

Schmidt [16] generalized Ash's work, giving an abstract characterization of RPFs on any finitely generated Fuchsian group of the first kind with parabolic elements, a class of groups which includes the Hecke groups. Schmidt [15] and Schmidt and Sheingorn [17] have taken steps toward an explicit characterization of RPFs on the Hecke groups using generalizations of the classical continued fractions and binary quadratic forms.

In this paper we give an explicit characterization of a class of rational period functions on the Hecke groups. Our characterization is for rational period functions of weight $2k$, with k a positive odd integer, and with irreducible pole

*1991 Mathematics Subject Classification. Primary 11F67; Secondary 11F12, 11E45.

sets which are Hecke-symmetric, as defined in Section 2.3. We also determine an explicit expression for all rational period functions on Hecke groups. We use the correspondence, developed in [5], between poles of rational period functions and generalized binary quadratic forms.

2 DEFINITIONS AND BACKGROUND IDEAS

2.1 Hecke groups

Let λ be a positive real number and let $G(\lambda) = \langle S, T\rangle/\{\pm I\}$, where $S = S_\lambda = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and I is the identity. Erich Hecke [8] showed that the only values of λ between 0 and 2 for which $G(\lambda)$ is discrete are $\lambda = \lambda_p = 2\cos(\pi/p)$ for $p = 3, 4, 5, \ldots$. These are the *Hecke groups* which we will denote by $G_p = G(\lambda_p)$ for $p \geq 3$. Put $U = U_{\lambda_p} = S_{\lambda_p} T \in G_p$. The Hecke groups have two relations, $T^2 = U^p = I$.

Hecke showed that $G(\lambda)$ is also discrete if $\lambda \geq 2$, however these groups have only one relation $T^2 = I$. In this case the related rational period functions have a simpler structure, which is given in [7].

For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_p$, we have $a, b, c, d \in \mathbb{Z}[\lambda_p]$ and $ad - bc = 1$, so G_p is a subgroup of $\mathrm{SL}(2, \mathbb{Z}[\lambda_p])$. It is well-known that $G_3 = \mathrm{SL}(2, \mathbb{Z}[\lambda_3])$ ($i.e.$,$\Gamma(1) = \mathrm{SL}(2, \mathbb{Z})$), however for the other Hecke groups $G_p \subsetneqq \mathrm{SL}(2, \mathbb{Z}[\lambda_p])$.

Members of Hecke groups act on the Riemann sphere as linear fractional transformations. An element $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_p$ is *hyperbolic* if $|a + d| > 2$, *parabolic* if $|a + d| = 2$, and *elliptic* if $|a + d| < 2$. We will designate fixed points accordingly.

We will use the following lemma when we study the poles of rational period functions.

Lemma 1. *Fix $p \geq 3$ and let $U = U_{\lambda_p}$. The nonzero entries of $U^n T \in G_p$ are all positive for $1 \leq n \leq p-1$. The only zero entries occur in $UT = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ and $U^{p-1}T = \left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)$.*

Proof. Write $U^n = \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)$. By induction on n we have that $a_n = \frac{sin((n+1)\pi/p)}{sin(\pi/p)}$ for $n \geq 0$, from which it is clear that $a_n > 0$ for $0 \leq n \leq p-2$. We write $U^n = UU^{n-1} = U^{n-1}U$, or

$$U^n = \begin{pmatrix} \lambda a_{n-1} - c_{n-1} & \lambda b_{n-1} - d_{n-1} \\ a_{n-1} & b_{n-1} \end{pmatrix} = \begin{pmatrix} \lambda a_{n-1} + b_{n-1} & -a_{n-1} \\ \lambda c_{n-1} + d_{n-1} & -c_{n-1} \end{pmatrix}.$$

Then

$$U^n = \begin{pmatrix} a_n & -a_{n-1} \\ a_{n-1} & -a_{n-2} \end{pmatrix},$$

so

$$U^n T = \begin{pmatrix} a_{n-1} & a_n \\ a_{n-2} & a_{n-1} \end{pmatrix},$$

which has positive entries for $2 \leq n \leq p-2$. $\square$

2.2 Rational period functions

For $M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{R})$ and $f(z)$ a complex function, we define the *weight $2k$ slash operator* $f \mid_{2k} M = f \mid M$ by

$$(f \mid M)(z) = (cz + d)^{-2k} f(Mz).$$

Definition 1. Let $U = U_{\lambda_p}$ for $p \geq 3$ and let $k \in \mathbb{Z}^+$. A *rational period function* (RPF) *of weight $2k$ for G_p* is a rational function $q(z)$ which satisfies the relations

$$q + q \mid T = 0, \tag{1}$$

and

$$q + q \mid U + q \mid U^2 + \cdots + q \mid U^{p-1} = 0. \tag{2}$$

Marvin Knopp [11] first introduced RPFs as period functions for automorphic integrals in the following way.

Definition 2. Let $S = S_{\lambda_p}$ for $p \geq 3$, and let $k \in \mathbb{Z}^+$. Suppose that F is a function meromorphic in $\mathcal{H}$ and at $i\infty$ which satisfies

$$(F \mid S)(z) = F(z),$$

and

$$(F \mid T)(z) = F(z) + q(z),$$

where $q(z)$ is a rational function. Then F is an *automorphic integral of weight $2k$ on G_p with rational period function $q(z)$.*

These definitions are equivalent. It is easy to show that any rational period function satisfying Definition 2 also satisfies Definition 1. On the other hand, Knopp showed [10, Theorem 3] that any function satisfying Definition 1 is the period function for an automorphic integral of weight $2k$.

For any rational period function q on G_p we let $P(q)$ denote the set of poles of q. Hawkins [6] introduced the idea of an *irreducible system of poles* (or *irreducible pole set*), the minimal set of poles which are forced to occur together by the relations (1) and (2). If $\alpha \in P^*(q) = P(q) \setminus \{0\}$, we let $P(q; \alpha)$ denote the irreducible set of poles of q which contains α.

2.3 Binary quadratic forms

An *indefinite binary quadratic form* on $\mathbb{Z}[\lambda_p]$ is an expression of the form

$$Q(x, y) = Ax^2 + Bxy + Cy^2,$$

with $A, B, C \in \mathbb{Z}[\lambda_p]$ and $D = B^2 - 4AC > 0$. We denote such a form by $Q = [A, B, C]$ and refer to it as a λ_p-BQF. If $\mathbb{Z}[\lambda_p]$ is a principal ideal domain we also require that a λ_p-BQF $[A, B, C]$ be *primitive*, *i.e.*, that $(A)+(B)+(C) = (1)$, where $(x) = x\mathbb{Z}[\lambda_p]$ denotes the ideal of $\mathbb{Z}[\lambda_p]$ generated by x.

We need a $1-1$ correspondence between λ_p-BQFs and certain algebraic numbers. We map the λ_p-BQF $Q = [A, B, C]$ to the number $\alpha_Q = \frac{-B+\sqrt{D}}{2A} \in \mathbb{Q}(\lambda_p, \sqrt{D})$. In this form the mapping is invertible only for $p = 3$, $G_p = \Gamma(1)$. When $p > 3$, ambiguity arises from the presence of nontrivial units in $\mathbb{Z}[\lambda_p]$ and from the fact that the discriminant D need not be square-free in $\mathbb{Z}[\lambda_p]$. In order to recover a $1-1$ correspondence from this mapping we first restrict the range of the mapping to hyperbolic fixed points of G_p. Then we give an algorithm which produces a unique inverse for any hyperbolic number.

Lemma 2. *If α is a hyperbolic fixed point of G_p it may be associated with a unique indefinite λ_p-BQF $Q = Q_\alpha$ such that $\alpha = \alpha_Q$. If $\mathbb{Z}[\lambda_p]$ is a principal ideal domain, Q may be chosen to be primitive.*

If an indefinite λ_p-BQF Q is associated with a hyperbolic number as in the Lemma, we say that Q is *hyperbolic*.

Proof. We present an outline of the more detailed proof in [5]. Suppose that α is a hyperbolic fixed point of G_p. Let $M_\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_p$ be a generator of the cyclic group of matrices fixing α. M_α is determined up to inverses and

$$\alpha = \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c}.$$

We show in [5] that if $\mathbb{Z}[\lambda_p]$ is a principal ideal domain, there is a uniquely determined positive number $g \in \mathbb{Z}[\lambda_p]$ with $(c, d-a, -b) = (g)$. Then $\frac{1}{g}[c, d - a, -b]$ is primitive and the unique primitive λ_p-BQF for α is

$$Q_\alpha = \begin{cases} \frac{1}{g}[c, d-a, -b], & \text{if } \alpha = \frac{a-d+\sqrt{(a+d)^2-4}}{2c}, \\ \frac{-1}{g}[c, d-a, -b], & \text{if } \alpha = \frac{a-d-\sqrt{(a+d)^2-4}}{2c}. \end{cases}$$

If $\mathbb{Z}[\lambda_p]$ fails to be a principal ideal domain we put $g = 1$. $\square$

Suppose that $\alpha = \frac{-B+\sqrt{D}}{2A}$ is a hyperbolic point associated with the λ_p-BQF $Q_\alpha = [A, B, C]$. We define the *Hecke* (λ_p-)*conjugate of* α to be $\alpha' = \frac{-B-\sqrt{D}}{2A}$, the number associated with $-Q_\alpha = [-A, -B, -C]$, *i.e.*, $\alpha'_Q = \alpha_{-Q}$. A calculation shows that if $V \in G_p$, then $(V\alpha)' = V\alpha'$. In the case of $G_3 = \Gamma(1)$, Hecke conjugation reduces to algebraic conjugation over $\mathbb{Q}$.

If R is a set of hyperbolic fixed points of G_p we write $R' = \{x' \mid x \in R\}$. We say that R has *Hecke* (λ_p-)*symmetry* if $R = R'$.

Elements of a Hecke group act on λ_p-BQFs by $(Q \circ M)(x, y) = Q(ax + by, cx + dy)$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_p$. This action preserves the discriminant and maps primitive forms to primitive forms. We say that λ_p-BQFs Q_1 and

Q_2 are G_p-*equivalent*, and write $Q_1 \sim Q_2$, if there exists a $V \in G_p$ such that $Q_2 = Q_1 \circ V$. It is easy to check that G_p-equivalence is an equivalence relation, so G_p partitions the λ_p-BQFs into equivalence classes of forms.

Every G_p-equivalence class of λ_p-BQFs contains either all hyperbolic forms or no hyperbolic forms [5]. Consequently, we may label equivalence classes themselves as hyperbolic or non-hyperbolic.

Suppose that Q_1 and Q_2 are hyperbolic λ_p-BQFs. A calculation shows that $Q_2 = Q_1 \circ V$ if and only if $\alpha_2 = V^{-1}\alpha_1$, where α_1 and α_2 are associated with Q_1 and Q_2, respectively. Thus G_p-equivalence of hyperbolic λ_p-BQFs induces a corresponding G_p-equivalence of associated numbers.

If $\mathcal{A}$ denotes a G_p-equivalence class of λ_p-BQFs, we let $-\mathcal{A} = \{-Q | Q \in \mathcal{A}\}$. Then $-\mathcal{A}$ is another G_p-equivalence class of forms, not necessarily distinct from $\mathcal{A}$. If $\mathcal{A}$ is hyperbolic, so is $-\mathcal{A}$, and the numbers associated with the forms in $-\mathcal{A}$ are the Hecke conjugates of the numbers associated with the forms in $\mathcal{A}$.

We say that a hyperbolic λ_p-BQF $Q = [A, B, C]$ is (G_p-)*simple* if $A > 0 > C$. If Q is G_p-simple we say that the associated hyperbolic number x_Q is (G_p-)*simple*. We show in [5] that a hyperbolic fixed point x is G_p-simple if and only if $x' < 0 < x$.

3 POLES OF RATIONAL PERIOD FUNCTIONS

Throughout this section we fix $p \geq 3$, $\lambda = \lambda_p$, and $U = U_{\lambda_p}$. We assume that q an RPF of weight $2k \in 2\mathbb{Z}^+$ on G_p with pole set $P = P(q)$.

3.1 Sets of Poles

In this section we show that the positive poles of q may be put into cycles, which we use to write the irreducible systems of poles. We also make the connection between these cycles of poles and simple λ_p-BQFs.

Lemma 3. *$P(q) \subset \mathbb{R}$ for any RPF q of weight $2k \in 2\mathbb{Z}^+$ on G_p.*

Remark. This is given without proof by Meier and Rosenberger [13], who state that the proof is analogous to the one by Knopp [12] for rational period functions on the modular group. The following proof is a generalization of the one by Choie and Zagier [4] for the modular group.

Proof. Suppose, by way of contradiction, that $\alpha_1 \in P$ but $\alpha_1 \notin \mathbb{R}$. By the first relation (1) we have $T\alpha_1 \in P$, then by the second relation (2) we have $U^{j_1}T\alpha_1 \in P$ for some j_1, $1 \leq j_1 \leq p-1$. Repeating this, we produce a sequence of poles $\{\alpha_1, \alpha_2, \ldots\}$, none of which are real, with $\alpha_{\nu+1} = U^{j_\nu}T\alpha_\nu$ and $1 \leq j_\nu \leq p-1$ for each $\nu \geq 1$. Lemma 1 and a geometric argument show that $|\arg(\alpha_\nu)| > |\arg(\alpha_{\nu+1})|$ for each $\nu \geq 1$. But this is impossible, since P is a finite set. $\square$

Put $\lambda = \lambda_p$ and $U = U_\lambda$ for $p \geq 3$. Define $\Phi_p : [0, \infty) \to [0, \infty)$ by

$$\Phi_p(x) = \begin{cases} TUx, & U^p(0) \leq x < U^{p-1}(0) \\ TU^2x, & U^{p-1}(0) \leq x < U^{p-2}(0) \\ \vdots & \\ TU^{p-1}x, & U^2(0) \leq x. \end{cases}$$

This function is given more explicitly by

$$\Phi_p(x) = \begin{cases} \frac{x}{1-\lambda x}, & 0 \leq x < 1/\lambda \\ \frac{1-\lambda x}{(\lambda^2-1)x-\lambda}, & 1/\lambda \leq x < \lambda/(\lambda^2-1) \\ \vdots & \\ x - \lambda, & \lambda \leq x. \end{cases}$$

It can be shown that $TU^n(x) > 0$ if and only if $U^{p-n+1}(0) < x < U^{p-n}(0)$, so the exponent n in $\Phi_p(x) = TU^n x$ is the *unique* exponent between 1 and $p-1$ for which $TU^n x > 0$ [5].

If $\mathcal{A}$ is a hyperbolic G_p-equivalence class of λ_p-BQFs we write $Z_{\mathcal{A}} = \{x \mid Q_x \in \mathcal{A}, Q_x \text{ simple}\}$. We show in [5] that each hyperbolic equivalence class of λ_p-BQFs contains at least one simple form, *i.e.*, $Z_{\mathcal{A}} \neq \emptyset$ for every hyperbolic $\mathcal{A}$. We also show in [5, Theorem 3] that the finite cyclic orbits of Φ_p are the sets $\{0\}$ and $Z_{\mathcal{A}}$, where $\mathcal{A}$ runs over all hyperbolic G_p-equivalence classes of λ_p-BQFs.

Let P^+ denote the set of positive poles in P, P^- denote the set of negative poles in P, so $P^* = P^+ \cup P^-$. Write $T\mathcal{Z}_{\mathcal{A}} = \{T\alpha \mid \alpha \in \mathcal{Z}_{\mathcal{A}}\}$.

The following Lemma establishes the relationship between irreducible pole sets of RPFs on G_p and equivalence classes of simple λ_p-BQFs.

Lemma 4. *Suppose that α is a positive pole of an RPF on G_p. Then α is hyperbolic and $\alpha \in Z_{\mathcal{A}}$, where $\mathcal{A}$ is the G_p-equivalence class of λ_p-BQFs containing Q_α. The irreducible pole set containing α is*

$$P(q; \alpha) = Z_{\mathcal{A}} \cup TZ_{\mathcal{A}}.$$

Proof. Suppose that $\alpha = \alpha_1 \in P^+$. As in the proof of Lemma 3, $\alpha_2 = U^{j_1}T\alpha_1 \in P$ for some j_1, $1 \leq j_1 \leq p-1$. By Lemma 1 we may take each entry of $U^{j_1}T$ to be non-negative, which implies that $\alpha_2 = U^{j_1}T\alpha_1 > 0$. Repeating this process gives a sequence of poles $\{\alpha_1, \alpha_2, \dots\} \subseteq P^+$, with

$$\alpha_{\nu+1} = U^{j_\nu}T\alpha_\nu, \tag{3}$$

and $1 \leq j_\nu \leq p-1$ for each $\nu \geq 1$. Since P^+ is finite we must have that

$$\alpha_1 = \alpha_{r+1} = U^{j_r}T\alpha_r, \tag{4}$$

for some $r \geq 1$. Thus we have a finite cycle of positive poles, $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$. Reversing (3) we have that $\alpha_r = TU^{p-j_r}\alpha_1 = \Phi_p(\alpha_{\nu+1})$, since $1 \leq p-j_\nu \leq p-1$

and $\alpha_\nu > 0$ for each $\nu \geq 1$. Reversing (4) we have that $\alpha_\nu = TU^{p-j_\nu}\alpha_{\nu+1} = \Phi_p(\alpha_1)$, since $1 \leq p - j_r \leq p-1$ and $\alpha_r > 0$. Hence $\{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subseteq P^+$ is a finite cyclic orbit of Φ_p, so $\{\alpha_1, \alpha_2, \ldots, \alpha_r\} = Z_{\mathcal{A}}$, where $\mathcal{A}$ is some G_p-equivalence class of λ_p-BQFs. Hence $\mathcal{A}$ is hyperbolic and contains Q_α, while Q_α and α are both hyperbolic and simple.

Since α is a positive pole, each element of $Z_{\mathcal{A}}$ must also be positive pole, so $Z_{\mathcal{A}} \subseteq P(q;\alpha)$. This, with the first relation (1), implies that every element of $TZ_{\mathcal{A}}$ is a negative pole, so $TZ_{\mathcal{A}} \subseteq P(q;\alpha)$. On the other hand, if $\beta \in P(q;\alpha)$, then $\beta = M\alpha$, where $M \in G_p$ is a product of matrices, each one equal to T or to U^i, $1 \leq i \leq p-1$. Thus α and β are G_p-equivalent numbers, hence Q_α and Q_β are G_p-equivalent BQFs, so $Q_\beta \in \mathcal{A}$. If $\beta > 0$, then $\beta \in Z_{\mathcal{A}}$. If $\beta < 0$, then $T\beta \in Z_{\mathcal{A}}$, and $\beta \in TZ_{\mathcal{A}}$. $\square$

Corollary. *The set of nonzero poles of an RPF on G_p has the form*

$$P^* = \bigcup_{\ell=1}^{L} \left(Z_{\mathcal{A}_\ell} \cup TZ_{\mathcal{A}_\ell}\right) \tag{5}$$

where the $\mathcal{A}_\ell$ are distinct indexed hyperbolic G_p-equivalence classes of λ_p-BQFs.

Remark. This is essentially Lemma 3.2 in [15].

The following Lemma allows us to express the negative poles in P^* using Hecke conjugation instead of the action of T.

Lemma 5. *Let $\mathcal{A}$ be a G_p-equivalence class of λ_p-BQFs. Then $TZ_{\mathcal{A}} = Z'_{-\mathcal{A}}$.*

Proof. A routine exercise showing containment in both directions establishes the Lemma. $\square$

3.2 Principal Parts

In this section we determine the principle part at any pole of any RPF of weight $2k$ on G_p. We use this to give an explicit expression of the form of any RPF.

3.2.1 The Pole at Zero

Lemma 6. *Suppose that q is an RPF of weight $2k$ on G_p. Then*

(i) q is regular at ∞,

(ii) if q has a pole at 0, the order of the pole is at most $2k$, and

(iii) q has a pole at 0 of order $2k$ if and only if $q(\infty) \neq 0$.

Proof. Fix $p \geq 3$ and put $\lambda = \lambda_p$ and $U = U_{\lambda_p}$. For *(i)* we apply $|S$ to the second relation (2), use the first relation (1) and rearrange to get

$$\left(q - q|S\right)(z) = \left(q|US + q|U^2S + \cdots + q|U^{p-2}S\right)(z).$$

We claim that the right hand side of this expression approaches 0 as $z \to \infty$. To prove our claim it suffices to show that for $1 \le n \le p-2$,

$$(q|U^n S)(z) \to 0 \qquad \text{as } z \to \infty.$$

Write $U^n = \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)$, so $U^n S = \left(\begin{smallmatrix} a_n & a_{n+1} \\ a_{n-1} & a_n \end{smallmatrix}\right)$ and

$$(q|U^n S)(z) = (a_{n-1}z - a_n)^{-2k}\, q\,(U^n S z)\,.$$

Now $(a_{n-1}z - a_n)^{-2k} \to 0$ as $z \to \infty$, since $a_{n-1} \ne 0$ for $1 \le n \le p-2$. We calculate that $0 < U^n S(\infty) < \infty$ for $1 \le n \le p-2$, and $U^n S(\infty)$ is parabolic, so $U^n S(\infty)$ cannot be a pole of q, *i.e.*, $q\,(U^n S z)$ is bounded near ∞. Thus $(a_{n-1}z - a_{n-2})^{-2k}\, q\,(U^n S z) \to 0$ as $z \to \infty$ $(1 \le n \le p-2)$, and the claim holds. If q had a pole at ∞ of order $m > 0$, then

$$q(z) = r(z) + c_m z^m + c_{m-1} z^{m-1} + \cdots + c_0,$$

where $r(\infty) = 0$ and $c_m \ne 0$. Then

$$(q - q|S)(z) = r(z) - r(z+\lambda) - m\lambda c_m z^{m-1} + \cdots\,,$$

where "$\cdots$" denotes a polynomial of degree less than $m-1$. Thus as $z \to \infty$,

$$(q - q|S)(z) \to \begin{cases} -\lambda c_1, & \text{if } m = 1, \\ \infty, & \text{if } m > 1, \end{cases}$$

which contradicts the claim and establishes *(i)*.

Parts *(ii)* and *(iii)* follow immediately from *(i)* and the first relation (1) in the form

$$q\left(\frac{-1}{z}\right) = -z^{2k} q(z).$$

□

An RPF of weight $2k$ on G_p may have a pole *only* at zero. Meier and Rosenberger show in [13] that such an RPF is of the form

$$q_{k,0}(z) = \begin{cases} a_0(1 - z^{-2k}), & \text{if } 2k \ne 2, \\ a_0(1 - z^{-2}) + b_1 z^{-1}, & \text{if } 2k = 2. \end{cases} \tag{6}$$

Given q, an RPF of weight $2k$ on G_p, we let $PP_\alpha[q]$ denote the principal part of $q(z)$ at $z = \alpha$. Then by (5) and Lemma 6, q has the form

$$q(z) = \sum_{\ell=1}^{L} \sum_{\alpha \in Z_{\mathcal{A}_\ell} \cup T Z_{\mathcal{A}_\ell}} PP_\alpha[q](z) + b_0 q_{k,0}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n}, \tag{7}$$

where $q_{k,0}$ is an RPF given by (6), b_0 and each a_n and c_n is a constant, and the $\mathcal{A}_\ell$ are indexed hyperbolic G_p-equivalence classes of λ_p-BQFs.

3.2.2 Nonzero Poles

Lemma 7. *Let q be an RPF of weight $2k$ on G_p with a nonzero pole at α. Then the pole at α is of order k.*

Remark. Schmidt [15] first states this Lemma with a note that the proof for RPFs on $\Gamma(1)$ goes through. The following proof generalizes the one for $\Gamma(1)$ by Choie and Zagier [4].

Proof. Fix $p \geq 3$ and put $U = U_{\lambda_p}$. Note that if $f(z)$ has a pole at β and $V = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ is a linear fractional transformation, then $(f \mid V)(z) = (cz+d)^{-2k} f(Vz)$ has a pole at $z = V^{-1}\beta$. In fact,

$$PP_{V^{-1}\beta}\left[f \mid V\right] = PP_{V^{-1}\beta}\left[PP_\beta[f] \mid V\right]. \tag{8}$$

By (1) and (8) with $V = T$, we have

$$PP_{T\beta}[q] = -PP_{T\beta}\left[PP_\beta[q] \mid T\right]. \tag{9}$$

In a similar way, we use (2) and (8) with $V = U^{-t}$, t an integer, to get

$$PP_{U^t\beta}[q] = -PP_{U^t\beta}\left[PP_\beta[q] \mid U^{-t}\right]. \tag{10}$$

Suppose that $\alpha > 0$. Then by the proof of Lemma 4, α is fixed by an element of G_p of the form $M = U^{j_r}TU^{j_{r-1}}T\ldots U^{j_1}T$, where $1 \leq j_\nu \leq p-1$ for each $\nu \geq 1$. Applying (9) and (10) r times each, we get

$$PP_{M\alpha}[q] = PP_{M\alpha}\left[PP_{M\alpha}[q] \mid M^{-1}\right],$$

and, since $M\alpha = \alpha$,

$$PP_\alpha[q] = PP_\alpha\left[PP_\alpha[q] \mid M^{-1}\right]. \tag{11}$$

If $\alpha < 0$, then $T\alpha \in P^+$ is fixed by an element of G_p of the form $N = U^{j_r}TU^{j_{r-1}}T\ldots U^{j_1}T$. Thus α is fixed by $M = TNT = TU^{j_r}TU^{j_{r-1}}T\ldots U^{j_1}$. We apply (9) and (10) r times each in this case as well, and q also satisfies (11) when $\alpha < 0$.

Let m be the order of the pole of q at α and suppose that q is normalized so that $PP_\alpha[q](z) = (z-\alpha)^{-m} + r_\alpha(z)$, where $r_\alpha(z)$ is a rational function with a pole at α of order less than m. Put $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ so $M^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. We calculate

$$\begin{aligned}
\left(PP_\alpha[q] \mid M^{-1}\right)(z) &= (-cz+a)^{-2k}\left(\left(M^{-1}z-\alpha\right)^{-m} + r_\alpha\left(M^{-1}z\right)\right) \\
&= (-cz+a)^{m-2k}\left((c\alpha+d)z-(a\alpha+b)\right)^{-m} + r_{\alpha,\frac{a}{c}}(z) \\
&= \frac{(-cz+a)^{m-2k}}{(c\alpha+d)^m}\left(z-M\alpha\right)^{-m} + r_{\alpha,\frac{a}{c}}(z) \\
&= \frac{(-c\alpha+a)^{m-2k}}{(c\alpha+d)^m}(z-\alpha)^{-m} + \tilde{r}_{\alpha,\frac{a}{c}}(z) \\
&= (c\alpha+d)^{2k-2m}(z-\alpha)^{-m} + \tilde{r}_{\alpha,\frac{a}{c}}(z),
\end{aligned}$$

where $r_{\alpha,\frac{a}{c}}$ and $\tilde{r}_{\alpha,\frac{a}{c}}$ are rational functions, each with a pole at α of order less than m and a pole at $\frac{a}{c}$. We have used partial fractions, the fact that M fixes α, and the fact that $(-c\alpha + a) = (c\alpha + d)^{-1}$. Then by (11) we have

$$\begin{aligned} PP_\alpha[q](z) &= PP_\alpha\left[(c\alpha+d)^{2k-2m}(z-\alpha)^{-m} + \tilde{r}_{\alpha,\frac{a}{c}}(z)\right] \\ &= (c\alpha+d)^{2k-2m}(z-\alpha)^{-m} + \tilde{r}_\alpha(z), \end{aligned}$$

where $\tilde{r}_\alpha$ is a rational function with a pole at α of order less than m. Thus $(c\alpha + d)^{2k-2m} = 1$, so either $m = k$ or $c\alpha + d = 1$. But if $c\alpha + d = 1$, then $\alpha = \frac{1-d}{c}$ is parabolic, a contradiction. We conclude that $m = k$. $\square$

Lemma 8. *Suppose that $\alpha \neq 0$ can occur as a pole of an RPF of weight $2k$ on G_p. Then there exists a unique function of the form*

$$q_{k,\alpha}(z) = \frac{1}{(z-\alpha)^k} + \frac{a_1(k,\alpha)}{(z-\alpha)^{k-1}} + \cdots + \frac{a_{k-1}(k,\alpha)}{z-\alpha},$$

such that for any *RPF q of weight $2k$ on G_p, $PP_\alpha[q]$ is a constant multiple of $q_{k,\alpha}$.*

Remark. The function $q_{k,\alpha}$ is uniquely determined by k and α, but is independent of the rational period function q.

Proof. Functions of the given form exist, since $PP_\alpha[q]$ has a pole of order k at α and can be normalized so that the coefficient of $(z-\alpha)^{-k}$ is 1. For uniqueness, suppose that $q_{k,\alpha}$ and $r_{k,\alpha}$ are functions satisfying the hypotheses of the Lemma, with $q_{k,\alpha} \neq r_{k,\alpha}$. Then there exist RPFs q and r and nonzero constants a and b such that $PP_\alpha[q] = aq_{k,\alpha}$ and $PP_\alpha[r] = br_{k,\alpha}$. But then $q - \frac{a}{b}r$ is an RPF with a pole at α of order less than $2k$, a contradiction. $\square$

The following Lemma gives a formula for $q_{k,\alpha}$.

Lemma 9. *Suppose that $\alpha \neq 0$ can occur as a pole of an RPF q of weight $2k$ on G_p. Then*

$$q_{k,\alpha}(z) = PP_\alpha\left[\frac{D^{k/2}}{Q_\alpha(z,1)^k}\right] = PP_\alpha\left[\frac{(\alpha-\alpha')^k}{(z-\alpha)^k(z-\alpha')^k}\right], \tag{12}$$

where Q_α is the λ_p-BQF associated to α, D is the discriminant of Q_α, and α' is the Hecke λ_p-conjugate of α.

Remark. Schmidt [15] states this Lemma in a modified form for $\alpha > 0$ and asserts that the proof follows from the proof in the classical case. The following proof holds for α positive or negative.

Proof. Write $Q_\alpha(z) = Q_\alpha(z,1) = Az^2 + Bz + C$. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of G_p which fixes both α and α'. We calculate

$$\begin{aligned}
&\left(Q_\alpha^{-k} \mid M^{-1}\right)(z) \\
&= (-cz+a)^{-2k} A^{-k} \left(\frac{dz-b}{-cz+a} - \alpha\right)^{-k} \left(\frac{dz-b}{-cz+a} - \alpha'\right)^{-k} \\
&= A^{-k}\left((c\alpha+d)z - (a\alpha+b)\right)^{-k}\left((c\alpha'+d)z - (a\alpha'+b)\right)^{-k} \\
&= A^{-k}(c\alpha+d)^{-k}(c\alpha'+d)^{-k}\left(z - \frac{a\alpha+b}{c\alpha+d}\right)^{-k}\left(z - \frac{a\alpha'+b}{c\alpha'+d}\right)^{-k} \\
&= A^{-k}(z-\alpha)^{-k}(z-\alpha')^{-k} \\
&= Q_\alpha(z)^{-k}.
\end{aligned}$$

We have used the fact that $(c\alpha+d)(c\alpha'+d) = 1$. From this it follows that Q_α^{-k} satisfies (11), as does q. By an argument similar to the proof of Lemma 8, $PP_\alpha[Q_\alpha^{-k}]$ is a nonzero multiple of $PP_\alpha[q]$, which is in turn a multiple of $q_{k,\alpha}$. Thus $q_{k,\alpha}$ is a multiple of $PP_\alpha[Q_\alpha^{-k}]$. Finally

$$\begin{aligned}
PP_\alpha[Q_\alpha^{-k}](z) &= PP_\alpha[A^{-k}(z-\alpha)^{-k}(z-\alpha')^{-k}] \\
&= A^{-k}(z-\alpha)^{-k}(\alpha-\alpha')^{-k} + \cdots \\
&= D^{-k/2}(z-\alpha)^{-k} + \cdots,
\end{aligned}$$

so the proportionality constants are as given in the Lemma. □

We now give a complete description of any RPF of weight $2k$ on G_p.

Theorem 1. *An RPF of weight $2k \in 2\mathbb{Z}$ on G_p is of the form*

$$q(z) = \sum_{\ell=1}^{L} C_\ell \sum_{\alpha \in Z_{\mathcal{A}_\ell}} \left(q_{k,\alpha}(z) - q_{k,T\alpha}(z)\right) + c_0 q_{k,0}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n},$$

where each $\mathcal{A}_\ell$ is a G_p-equivalence class of $\mathbb{Z}[\lambda_p]$-BQFs, $Z_{\mathcal{A}_\ell}$ is the cycle of positive poles associated with $\mathcal{A}_\ell$, $q_{k,\alpha}$ is given by (12)*, $q_{k,0}$ is given by* (6)*, and the C_ℓ and c_n are all constants.*

Proof. By (7) and Lemma 8 we have that any RPF q of weight $2k$ on G_p has the form

$$q(z) = \sum_{\ell=1}^{L} \sum_{\alpha \in Z_{\mathcal{A}_\ell} \cup TZ_{\mathcal{A}_\ell}} C_\alpha q_{k,\alpha}(z) + c_0 q_{k,0}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n},$$

where each $\mathcal{A}_\ell$ is a hyperbolic G_p-equivalence class of λ_p-BQFs and $q_{k,0}$ is given by (6). From (9) and (10) we see that the coefficients C_α alternate in sign as α goes through any cycle $\left\{\alpha_1, T\alpha_1, U^{j_1}T\alpha_1, TU^{j_1}T\alpha_1, \ldots\right\}$. Thus for each irreducible system of poles $Z_{\mathcal{A}_\ell} \cup TZ_{\mathcal{A}_\ell}$ there is a constant C_ℓ such that $C_\alpha = C_\ell$ for each $\alpha \in Z_{\mathcal{A}_\ell}$ and $C_\alpha = -C_\ell$ for each $\alpha \in TZ_{\mathcal{A}_\ell}$. As a result any RPF of weight $2k$ on G_p has the given form. □

Lemma 5 allows us to identify negative poles using Hecke conjugation instead of the action of T in the Corollary.

Corollary. *An RPF of weight $2k \in 2\mathbb{Z}$ on G_p is of the form*

$$q(z) = \sum_{\ell=1}^{L} C_\ell \left(\sum_{\alpha \in Z_{\mathcal{A}_\ell}} q_{k,\alpha}(z) - \sum_{\alpha \in Z_{-\mathcal{A}_\ell}} q_{k,\alpha'}(z) \right) + c_0 q_{k,0}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n}, \quad (13)$$

where each $\mathcal{A}_\ell$ is a G_p-equivalence class of $\mathbb{Z}[\lambda_p]$-BQFs, $Z_{\mathcal{A}_\ell}$ is the cycle of positive poles associated with $\mathcal{A}_\ell$, $q_{k,\alpha}$ is given by (12)*, $q_{k,0}$ is given by* (6)*, and the C_ℓ and c_n are all constants.*

4 HECKE-SYMMETRIC POLE SETS, k ODD

The following theorem characterizes of RPFs of weight $2k$ on G_p, for k odd, with Hecke-symmetric sets of poles.

Theorem 2. *Suppose that k is odd.*

Suppose q is an RPF of weight $2k$ on G_p with Hecke λ_p-symmetric irreducible systems of poles. Then q is of the form

$$q(z) = \sum_{\ell=1}^{L} d_\ell \sum_{\alpha \in Z_{\mathcal{A}_\ell}} Q_\alpha(z,1)^{-k} + c_0 q_{k,0}(z), \quad (14)$$

where each $\mathcal{A}_\ell$ is a G_p-equivalence class of $\mathbb{Z}[\lambda_p]$-BQFs satisfying $-\mathcal{A}_\ell = \mathcal{A}_\ell$, the d_ℓ $(1 \le \ell \le M)$ are constants, and $q_{k,0}(z)$ is given by (6)*.*

Conversely, any rational function of the form (14) *is an RPF of weight $2k$ on G_p with Hecke λ_p-symmetric irreducible systems of poles.*

Proof. Suppose that q is an RPF of weight $2k$ on G_p with Hecke λ_p-symmetric irreducible systems of poles. Then $-\mathcal{A} = \mathcal{A}$ for each G_p-equivalence class of λ_p-BQFs associated with poles of q. Thus the expression for q in (13) simplifies to

$$q(z) = \sum_{\ell=1}^{L} C_\ell \sum_{\alpha \in Z_{\mathcal{A}_\ell}} (q_{k,\alpha}(z) - q_{k,\alpha'}(z)) + c_0 q_{k,0}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n},$$

where $q_{k,0}(z)$ is given by (6) and C_ℓ $(1 \le \ell \le L)$ and c_n $(1 \le n \le 2k-1)$ are all constants. A calculation shows that for any pole α,

$$q_{k,\alpha}(z) - q_{k,\alpha'}(z) = PP_\alpha \left[\frac{D^{k/2}}{Q_\alpha(z,1)^k} \right] - (-1)^k PP_{\alpha'} \left[\frac{D^{k/2}}{Q_\alpha(z,1)^k} \right],$$

where D is the discriminant of Q_α. Since k is odd,

$$q_{k,\alpha}(z) - q_{k,\alpha'}(z) = \frac{D^{k/2}}{Q_\alpha(z,1)^k},$$

so

$$q(z) = \sum_{\ell=1}^{L} d_\ell \sum_{\alpha \in Z_{\mathcal{A}_\ell}} Q_\alpha(z,1)^{-k} + c_0 q_{k,0}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n}, \tag{15}$$

where $d_\ell = C_\ell D_\ell^{k/2}$ and D_ℓ is the discriminant of λ_p-BQFs in $\mathcal{A}_\ell$.

Schmidt proves in [15, Theorem 3.1] that if k is odd, then

$$r(z) = \sum_{\alpha \in Z_{\mathcal{A}} \cup Z_{-\mathcal{A}}} Q_\alpha(z,1)^{-k}$$

is an RPF of weight $2k$ on G_p. Thus the first sum in (15) is an RPF of weight $2k$ on G_p. Since q and $q_{k,0}$ are also RPFs, we must have that $\sum_{n=1}^{2k-1} \frac{c_n}{z^n}$ is an RPF of weight $2k$ on G_p. But then we must have $c_n = 0$ for $1 \le n \le 2k-1$ or else we contradict Lemma 6 (*ii*). Thus q has the form (14).

For the converse, assume that q has the form (14). Since $\sum_{\ell=1}^{L} d_\ell \sum_{\alpha \in Z_{\mathcal{A}_\ell}} Q_\alpha(z,1)^{-k}$ and $q_{k,0}$ are both RPFs of weight $2k$ on G_p, q is also an RPF of weight $2k$ on G_p. The set of nonzero poles of q is

$$P^*(q) = \bigcup_{\ell=1}^{L} \left(Z_{\mathcal{A}_\ell} \cup Z'_{\mathcal{A}_\ell} \right),$$

which is clearly Hecke-symmetric.

□

References

[1] A. Ash, "Parabolic cohomology of arithmetic subgroups of $SL(2,\mathbf{Z})$ with coefficients in the field of rational functions on the Riemann sphere," *Amer. J. Math.* **111** (1989), 35–51.

[2] Y. Choie and L. A. Parson, "Rational period functions and indefinite binary quadratic forms. I," *Math. Ann.* **286** (1990), 697–707.

[3] Y. Choie and L. A. Parson, "Rational period functions and indefinite binary quadratic forms, II," *Illinois J. Math.* **35** (1991), 374–400.

[4] Y. Choie and D. Zagier, "Rational period functions for $PSL(2,\mathbf{Z})$," in [9], 1993, pp. 89–108.

[5] W. Culp-Ressler, "On binary quadratic forms and the Hecke groups," preprint.

[6] J. H Hawkins, "On rational period functions for the modular group," Unpublished manuscript.

[7] J. H. Hawkins and M. I. Knopp, "A Hecke correspondence theorem for automorphic integrals with rational period functions," *Illinois J. Math.* **36** (1992), 178–207.

[8] E. Hecke, "Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung," *Math. Ann.* **112** (1936), 664–699.

[9] M. Knopp and M. Sheingorn (eds.), *A tribute to Emil Grosswald: Number theory and related analysis*, Vol. 143 of *Contemporary Mathematics Series*, American Mathematical Society, Providence, 1993.

[10] M. I. Knopp, "Some new results on the Eichler cohomology of automorphic forms," *Bull. Amer. Math. Soc. (N.S.)* **80** (1974), 607–632.

[11] M. I. Knopp, "Rational period functions of the modular group," *Duke Math. J.* **45** (1978), 47–62.

[12] M. I. Knopp, "Rational period functions of the modular group II," *Glasgow Math. J.* **22** (1981), 185–197.

[13] H. Meier and G. Rosenberger, "Hecke-Integrale mit rationalen periodischen Funktionen und Dirichlet-Reihen mit Funktionalgleichung," *Resultate Math.* **7** (1984), 209–233.

[14] L. A. Parson, "Rational period functions and indefinite binary quadratic forms, III," in [9], 1993, pp. 109–116.

[15] T. A. Schmidt, "Remarks on the Rosen λ-Continued Fractions," in *Number theory with an emphasis on the Markoff spectrum* (A. Pollington and W. Moran, eds.), Vol. 147 of *Lecture notes in pure and applied mathematics*, New York, pp. 227–238, 1993.

[16] T. A. Schmidt, "Rational period functions and parabolic cohomology," *J. Number Theory* **57** (1996), 50–65.

[17] T. A. Schmidt and M. Sheingorn, "Length spectra of the Hecke triangle groups," *Math. Z.* **220** (1995), 369–397.